ON THE MARTINGALE-FAIR INDEX OF RETURN FOR INVESTMENT FUNDS

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Abstract. A concept of martingale-fair index of return, consistent with Arbitrage Free Pricing Theory, is introduced. An explicit formula for the average rate of return of a group of investment/pension funds in a discrete time stochastic model is derived and several properties of this index are shown. In particular, it is proven to be martingale-fair, i.e. be a martingale provided the prices of assets on the financial market form a vector martingale. The problem of merger of the funds is treated in detail.

1. Introduction

Statistical indexes play a very important role in both economic theory and practice for they simply aggregate information coming from the economy to every decision maker. There are two modern approaches to selecting an index number formula. The axiomatic approach — dating back to Fischer (1922) — focuses on a desired perfomance of the index in response to particular types of changes. The alternative approach, including e.g. cost of living index theory, concerns index's ability to reflect a substitution behavior on the part of economic agents.

This paper concerns both approaches applied to selecting an index formula which would reflect accurately investment results of investment pension

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funds. We investigate an index formula which accounts for the rate of return achieved by the group of funds as well as satisfies several axioms the most important of which is the martigale property. This axiom is based on the Arbitrage Free Pricing Theory due to S. Ross (see [22]). According to his Fundamental Theorem of Asset Pricing the securities market is arbitrage free if and only if there exists a risk neutral probability measure (see Panjer et al (1998) p. 211). But then the discounted value of the portfolio corresponding to a self-financing strategy is a martigale under a risk-neutral probability measure (see e.g. Proposition 5.3.8 of Panjer et al (1998)). This shows a central role of martingales in modelling securites market as well as the importance of the notion of martingale itself.

An important consequence of the Fundamental Theorem of Asset Pricing is that any index properly reflecting investment efficiency of the group of investment funds should behave like a martingale if the whole market value is a martingale. Throughout the paper we call this property the *martingale fairness axiom* (shortly, the fairness axiom) and say shortly that the index is *fair*. To enlight the name of the axiom, recall that a random game with two players is called fair if the expectation of its pay off is zero. Martingale fairness means that a multiperiod game has, after each period, a pay off with a conditional expectation zero subject to a given result from the previous period. Thus it may be specially useful in testing stochastic dynamics of financial market indexes.

The problem of measuring investment results is also of great practical importance. In several countries statistical indexes of such kind are used as benchmarks for privately managed pension funds. Using “naive” indexes may lead then to overestimation of the market investment performance increasing the risk of an extra contribution to the fund’s assets by this assets management company which operates the fund performing below the benchmark (see e.g. [16]). Thus the benchmark should be fair in the sense that it reflects properly the market performance. The martingale — fairness axiom,

formulated in this paper, formalizes that intuition. Some other arguments to apply carefully chosen investment benchmarks for pension funds can be found in [2].

To be more concrete, let us consider a group of n investment funds. Let $k_i(t)$ and $w_i(t)$ denote a number of all accounting units possessed by the members of the i-th investment fund and a value of the i-th fund unit at time t, respectively. The problem considered in this paper is how to define an average rate of return to meet the fairness axiom as well as several other requirements in a discrete time stochastic model. In Section 2 we develop a simple stochastic model with discrete time. This allows us to derive a proper definition of the average rate of return. Our definition is as follows

$$\bar{r}_A(s,t) = \prod_{u=s}^{t-1} \left(1 + \frac{\sum_{i=1}^{n} r_i(u,u+1) k_i(u) w_i(u)}{\sum_{i=1}^{n} k_i(u) w_i(u)} \right) - 1. \tag{1.1}$$

This model is also a starting point to a more involved continuous time stochastic model of funds dynamics presented in Gajek and Kałuszka(2004).

The paper sheds a new light on the problem of constructing index numbers. The axiomatic approach, exemplified by Irving Fisher's *The Making of Index Numbers* (1922), evaluates indexes based on whether they perform as desired when prices or quantities undergo particular types of changes. Eichorn and Voeller (1976) provided a discussion of the "test" approach to index number theory. Bulk (1995) gave a survey of axiomatic price index theory (a comprehensive account of the price index was given by Afriat (1977)). Fisher and Shell (1972, 1992) surveyed developments in statistical and economic index numbers and in productivity measurement; see also Eichorn el al. (1978), Diewert (1976, 1978), Balk and Diewert (2001) and Dumagan (2002). The stochastic approach to the construction of price index numbers was presented by Ogwang (1995) (see also the papers cited therein). Comparing with the above authors, we introduce *martingale fairness* — a new axiom to test the desired performance of the rate of return index in response to martingale-type changes of the asset prices. The axiom concerns

stochastic approach to the market dynamics, is mathematically consistent with Arbitrage Free Pricing Theory and has appropriate economic implications.

2. Average rate of return in a discrete-time model

Throughout this Section we assume that $k_i(t)$ and $w_i(t)$ are observed in discrete moments $t = 0, 1, 2, \ldots$. The problem is to find a proper definition of the average return for a group of n investment funds during a given time period $[s, t]$. Let us denote it by $\bar{r}_A(s, t)$. Clearly, $\bar{r}_A(s, t)$ should take values in $(-1, \infty)$.

2.1. **The model.** We will use the following state-variables:

$c_j(t)$ = price of asset j of the financial market at time t, $j = 1, \ldots, N$,

$u_{ij}(t)$ = number of units of asset j possessed by the i-th fund at time t, $i = 1, \ldots, n$, $j = 1, \ldots, N$,

$w_i(t)$ = value of participation unit of the i-th fund at time t,

$k_i(t)$ = number of units of the i-th fund at time t,

$d_i(t)$ = the amount of contributions of the i-th fund's liabilities minus the drawdown amount of i-th fund's liabilities at time t,

$d(t) = \sum_{i=1}^n d_i(t)$,

$A_i(t)$ = value of i-th fund's assets,

$A(t) = \sum_{i=1}^n A_i(t)$,

$A_i^*(t) = A_i(t)/A(t)$ — a relative value of the assets of the i-th fund at time t,

$\Delta f(x) = f(x + 1) - f(x+)$.

We will assume that:

1) All investments are infinitely divisible.

2) There are no transaction costs or taxes and the assets pay no dividends.

3) Member does not pay for allocation of his/her wealth.

4) No consumption of funds exists.

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{F} = \{\mathcal{F}_0,\ \mathcal{F}_1,\ \mathcal{F}_2, \dots\}$ be a filtration, i.e. a stream of σ-algebras of subsets of Ω with the property $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \subseteq \mathcal{F}$. Without loss of generality, we may put $\mathcal{F}_0 = \{\emptyset, \Omega\}$. The filtration $\mathbb{F}$ describes the process how information is revealed to the investors. We will assume that $c_i(t)$ is measurable with respect to $\mathcal{F}_t$ (written $\mathcal{F}_t$-measurable) for each i, t. Given t, we have

$$w_i(t)k_i(t) = u_{i1}(t)c_1(t) + \ldots + u_{iN}(t)c_N(t), \quad i = 1, \ldots n. \tag{2.1}$$

Here and subsequently, the symbol $X = Y$ means that the random variables X, Y are defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{P}(X = Y) = 1$. We assume that each random variable $w_i(t)$ is adapted to $\mathbb{F}$ means that $w_i(t)$ is $\mathcal{F}_t$-measurable for each i, t. We also assume that both $k_i(t)$ and $u_{ij}(t)$ are adapted to $\mathbb{F}$. It means that we are allowing the investor to buy (sell) units after the values $c_i(t)$ are observed.

At any time t, split of units is allowed. The new price of one unit and the number of units of i-th fund are denoted by $w_i(t+)$ and $k_i(t+)$, respectively. At time t we have

$$w_i(t+)k_i(t+) = w_i(t)k_i(t), \quad i = 1, \ldots, n, \tag{2.2}$$

and at the time $t + 1$,

$$w_i(t+1)k_i(t+) = u_{i1}(t)c_1(t+1) + \ldots + u_{iN}(t)c_N(t+1), \qquad i = 1, \ldots, n. \tag{2.3}$$

From (2.1)–(2.3) we get

$$k_i(t+)(w_i(t+1) - w_i(t+)) = \sum_{j=1}^{N} u_{ij}(t)(c_j(t+1) - c_j(t)) \tag{2.4}$$

for $i = 1, \ldots, n$. Moreover, any member of the i-th fund reallocate his/her wealth. The members withdraw $k_i^{(W)}(t+1)w_i(t+1)$ of monetary units from the i-th fund and invest $k_j^{(I)}(t+1)w_j(t+1)$ of the amount $\sum_{i=1}^{n} k_i^{(W)}(t +$

$1)w_i(t+1)$ in the j-th fund, where $k_i^{(W)}(t+1)$ and $k_i^{(I)}(t+1)$ are $\mathcal{F}_{t+1}$-measurable random variables for each i. At time $t+1$, the stream of liability payments also changes balance of the i-th fund. As a consequence the number of units of the i-th fund changes from $k_i(t+)$ to $k_i(t+1)$:

$$\begin{aligned} k_i(t+1)w_i(t+1) &= k_i(t+)w_i(t+1) - k_i^{(W)}(t+1)w_i(t+1) + \\ &+ k_j^{(I)}(t+1)w_i(t+1) + d_i(t+1), \quad i = 1, 2, \ldots, n, \end{aligned} \tag{2.5}$$

where $d_i(t+1)$ is $\mathcal{F}_{t+1}$-measurable each i, t. After summing equations in (2.5), we get the following simple equation

$$\sum_{i=1}^{n} w_i(t+1)(k_i(t+1) - k_i(t+)) = d(t+1). \tag{2.6}$$

After allocation of clients, the management of the i-th fund rebalances the portfolio:

$$\begin{aligned} w_i(t+1)k_i(t+1) =& u_{i1}(t+1)c_1(t+1) + \ldots + \\ &+ u_{iN}(t+1)c_N(t+1) \end{aligned} \tag{2.7}$$

for $i = 1, \ldots, n$.

To conclude, our mathematical model of dynamics of a group of funds leads to the following stochastic difference equations:

$$w_i(t)k_i(t) = \sum_{j=1}^{N} u_{ij}(t)c_j(t), \tag{2.8}$$

$$w_i(t+)k_i(t+) = w_i(t)k_i(t), \tag{2.9}$$

$$k_i(t+)\Delta w_i(t) = \sum_{j=1}^{N} u_{ij}(t)\Delta c_j(t), \tag{2.10}$$

$$\begin{aligned} w_i(t+1)\Delta k_i(t) =& \left(k_i^{(I)}(t+1) - k_i^{(W)}(t+1)\right) w_i(t+1) + \\ &+ d_i(t+1), \end{aligned} \tag{2.11}$$

$$w_i(t+1)\Delta k_i(t) = \sum_{j=1}^{N} c_j(t+1)\Delta u_{ij}(t), \tag{2.12}$$

where $t = 0, 1, 2 \dots$ and $i = 1, 2, \dots, n$. In this model functions u_{ij}, $k_i^{(I)}$ and $k_i^{(W)}$ play a role of control variables.

2.2. **Definition of average return.** Our definition of the average return in discrete-time model in the time period $[s, t]$ is given by the formula

$$\bar{r}_A(s,t) = \prod_{u=s}^{t-1} \big(1 + \sum_{i=1}^{n} A_i^*(u) r_i(u, u+1)\big) - 1, \tag{2.13}$$

where $r_i(u, u+1)$ is the return of the i-th fund in the time interval $(u, u+1]$, i.e.

$$r_i(u, u+1) = \frac{w_i(u+1) - w_i(u+)}{w_i(u+)}. \tag{2.14}$$

For convenience, we put $\bar{r}_A(s, s) = 0$ for each s.

We give two main arguments for using definition (2.13). The first one is based on analysis of changes of total assets of the funds. Observe that

$$\frac{A(t) - A(s)}{A(s)} = \prod_{u=s}^{t-1} \left(1 + \frac{A(u+1) - A(u)}{A(u)}\right) - 1.$$

Applying (2.2) and (2.6) we get

$$A(u+1) - A(u) = \sum_{i=1}^{n} w_i(u+1) k_i(u+1) - \sum_{i=1}^{n} w_i(u) k_i(u) =$$
$$= \sum_{i=1}^{n} w_i(u+1) k_i(u+1) - \sum_{i=1}^{n} w_i(u+) k_i(u+) =$$
$$= \sum_{i=1}^{n} w_i(u+1)(k_i(u+1) - k_i(u+)) + \sum_{i=1}^{n} (w_i(u+1) - w_i(u+)) k_i(u+) =$$
$$= d(u+1) + \sum_{i=1}^{n} A_i(u) \frac{w_i(u+1) - w_i(u+)}{w_i(u+)}.$$

Hence

$$\frac{A(t) - A(s)}{A(s)} = \prod_{u=s}^{t-1} \left(1 + \frac{d(u+1)}{A(u)} + \sum_{i=1}^{n} A_i^*(u) r_i(u, u+1)\right) - 1.$$

After removing the influence of the amount of contribution on fund's liabilities, we arrive at

$$\frac{A(t)-A(s)}{A(s)}=\bar{r}_A(s,t).$$

The second argument is as follows. Clearly

$$\bar{r}_A(t,t+1)=\sum_{j=1}^{n}A_j^*(t)r_j(t,t+1).$$

Since $A_j^*(t)\geq 0$ and $\sum_{j=1}^{n}A_j^*(t)=1$, we have the following interpretation of the average return: $\bar{r}_A(t,t+1)$ is equal to the expected return of K_0 monetary units chosen at random from the assets of the group of funds at time t, i.e.

$$\bar{r}_A(t,t+1)=\mathbb{E}r_J(t,t+1),$$

where J is a random variable with the distribution $\mathbb{P}(J=j)=A_j^*(t)$, $j=1,\dots,n$. Moreover, for every $s<t$,

$$\begin{aligned}\bar{r}_A(s,t)=&\prod_{u=s}^{t-1}\big((1+\mathbb{E}r_{J_u}(u,u+1)\big)-1=\\=&\mathbb{E}\prod_{u=s}^{t-1}\big((1+r_{J_u}(u,u+1)\big)-1,\end{aligned}$$

where $J_s,\dots,J_{t-1}$ are independent random variables such that $\mathbb{P}(J_u=j)=A_j^*(u)$, $j=1,\dots,n$, $u=0,1,2,\dots$. This means that if we repeat the procedure of choosing independently and sequentially K_0 monetary units from the assets of one fund at time $s,s+1,\dots,t-1$ (with reinvestment at funds), then our capital at time t will be a random variable K with

$$\mathbb{E}K=K_0(1+\bar{r}_A(s,t)).$$

In other words, the expected value of rate of return is equal to the average rate of return:

$$\mathbb{E}\left(\frac{K-K_0}{K_0}\right)=\bar{r}_A(s,t).$$

Remark 2.1. After writting the paper we found out that the definition (2.13) appears in a quite different context in Barber et al (2001), but we have not found so far any analysis justifying the use of it as a definition of the average rate of return.

3. Martingale fairness and other properties of the average rate of return $\bar{r}_A$

The main result of the paper is the following theorem.

Theorem 3.1. *The index number $\bar{r}_A$ is fair, i.e. if $\{c_i(t),\ t = 0, 1, 2, \dots\}$ is an $\mathbb{F}$-martingale for each i, then $\{\bar{r}_A(0,t),\ t = 0, 1, 2, \dots\}$ is an $\mathbb{F}$-martingale. Moreover, if $\{c_i(t),\ t = 0, 1, 2, \dots\}$ is an $\mathbb{F}$-submartingale (resp. $\mathbb{F}$-supermartingale) for each i, then $\{\bar{r}_A(0,t),\ t = 0, 1, 2, \dots\}$ is an $\mathbb{F}$-submartingale (resp. $\mathbb{F}$-supermartingale).*

Proof. By definition, the random variable $\bar{r}_A(0,t)$ is $\mathcal{F}_t$-measurable for each t. By assumption both $k_i(t)$ and $w_i(t)$ are $\mathcal{F}_t$-measurable. Moreover, $A_i^*(t)$ is $\mathcal{F}_t$-measurable, since

$$A_i^*(t) = \frac{w_i(t)k_i(t)}{\sum_{i=1}^n w_i(t)k_i(t)}.$$

Of course $\sum_{j=1}^n A_j^*(t) = 1$ and we have

$$\mathbb{E}\left(\bar{r}_A(0,t+1) \mid \mathcal{F}_t\right) = (\bar{r}_A(0,t)+1)\mathbb{E}\left[\sum_{i=1}^n A_i^*(t)\frac{w_i(t+1)}{w_i(t+)} \mid \mathcal{F}_t\right] - 1 =$$

$$= (\bar{r}_A(0,t)+1)\left(\sum_{i=1}^n A_i^*(t)\mathbb{E}\left[\frac{w_i(t+1)}{w_i(t+)} \mid \mathcal{F}_t\right]\right) - 1.$$

We show that

$$\mathbb{E}\left[\frac{w_i(t+1)}{w_i(t+)} \mid \mathcal{F}_t\right] = 1$$

for each i. Recall that

$$w_i(t+1)k_i(t+) = u_{i1}(t)c_1(t+1) + \ldots + u_{iN}(t)c_N(t+1),$$

$$w_i(t+)k_i(t+) = w_i(t)k_i(t) = u_{i1}(t)c_1(t) + \ldots + u_{iN}(t)c_N(t),$$

for $i = 1, \dots, n$. Since $c(t)$ is an $\mathbb{F}$-martingale, we get

$$\mathbb{E}\left(\frac{w_i(t+1)}{w_i(t+)} \mid \mathcal{F}_t\right) = \frac{1}{k_i(t+)w_i(t+)}\mathbb{E}\left(\sum_{j=1}^{N} u_{ij}(t)c_j(t+1) \mid \mathcal{F}_t\right) =$$

$$= \frac{1}{k_i(t+)w_i(t+)}\sum_{j=1}^{N} u_{ij}(t)\mathbb{E}(c_j(t+1) \mid \mathcal{F}_t) =$$

$$= \frac{1}{k_i(t+)w_i(t+)}\sum_{j=1}^{N} u_{ij}(t)c_j(t) = \frac{k_i(t)w_i(t)}{k_i(t+)w_i(t+)} = 1.$$

The proof of the first part of the theorem is completed. The proof of the second part is analogous so it is omitted. □

Remark 3.1. A natural question arises if other indexes of the average rate of return are martingale-fair. An interesting example is provided by the Polish law regulations on operation of pension funds (see The Law on Organisation and Operation of Pension Funds, Art. 173, Dziennik Ustaw Nr 139 poz. 934, Art. 173; for the English translation see *Polish Pension ...*, 1997). The following definition of the average return of a group of pension funds can be found there:

$$\bar{r}_{PL}(s,t) = \sum_{i=1}^{n} \frac{1}{2} r_i(s,t)\left(\frac{w_i(s)k_i(s)}{\sum_{j=1}^{n} w_j(s)k_j(s)} + \right.$$
$$\left. + \frac{w_i(t)k_i(t)}{\sum_{j=1}^{n} w_j(t)k_j(t)}\right), \tag{3.1}$$

where $r_i(s,t)$ denotes the rate of return of the i-th fund, that is,

$$r_i(s,t) = \frac{w_i(t) - w_i(s)}{w_i(s)}.$$

The average rate of return defined above is not martingale-fair in general. In fact, assume that $k_i(s) = k$, $w_i(0) = 1$, and $u_{ij}(s) = u_{ij}$ with $k, u_{ij} \in \mathbb{R}$ for each i, j, s. After an elementary algebra we get

$$\bar{r}_{PL}(0,t) = \frac{1}{2n}\sum_{i=1}^{n} w_i(t) - 1 + \frac{1}{2}\frac{\sum_{i=1}^{n}(w_i(t))^2}{\sum_{i=1}^{n} w_i(t)},$$

where $w_i(t) = (u_{i1}c_1(t) + \ldots + u_{iN}c_N(t))/k$. Since $(w_1 + \ldots + w_n)^2 \leq n(w_1^2 + \ldots + w_n^2)$ for all reals w_i, we have

$$\begin{aligned}\mathbb{E}\bar{r}_{PL}(0,t) =& \frac{1}{2n}\sum_{i=1}^{n}\mathbb{E}w_i(t) - 1 + \frac{1}{2}\mathbb{E}\left(\frac{\sum_{i=1}^{n}(w_i(t))^2}{\sum_{i=1}^{n}w_i(t)}\right) \geq \\ &\geq -\frac{1}{2} + \frac{1}{2n}\mathbb{E}\sum_{i=1}^{n}w_i(t) = 0 = \mathbb{E}\bar{r}_{PL}(0,0), \end{aligned} \tag{3.2}$$

and equality holds in (3.2) if and only if $\mathbb{P}(w_1(t) = \ldots = w_n(t)) = 1$.

Suppose that $u_{ik} \neq u_{jk}$ for some i, j, k, and suppose $c_1(t), \ldots, c_N(t)$ are not linearly dependent, i.e. for any reals $a_1, \ldots, a_N$ such that $\mid a_1 \mid + \ldots + \mid a_N \mid \neq 0$,

$$\mathbb{P}(a_1c_1(t) + \ldots + a_Nc_N(t) = 0) < 1.$$

Then $\mathbb{P}(w_1(t) = \ldots = w_n(t)) < 1$ and

$$\mathbb{E}\bar{r}_{PL}(0,t) > \mathbb{E}\bar{r}_{PL}(0,0).$$

This means that $\{\bar{r}_{PL}(0,t),\ t = 0, 1, 2, \ldots\}$ cannot be a martingale.

Now, we formulate several axioms which any properly defined average rate of return should satisfy. Some of them can be found in Kellison(1991). The average return $\bar{r}_A$ meets all the demands. The proofs are straightforward so they will be omitted.

Property 3.1. *If the group consists only of the i-th fund, then*

$$\bar{r}_A(s,t) = \frac{w_i(t) - w_i(s+)}{w_i(s+)}$$

Property 3.2 (Multiplication axiom)**.** *For every $s < u < t$*

$$1 + \bar{r}_A(s,t) = (1 + \bar{r}_A(s,u))(1 + \bar{r}_A(u,t))$$

The next property says that the average rate of return $\bar{r}_A$ is consistent with respect to grouping of funds.

Property 3.3 (Consistency in aggregation-axiom)**.** *If funds are grouped, and if the average rate of return of every group is calculated over the time interval* $[t, t+1)$*, then the average rate of return of groups equals to the average rate of return of all funds over the time interval* $[s, s+1)$*.*

It is easy to check that $\bar{r}_{PL}$ does not possess Properties 3.2 and 3.3. For instance, we show that Property 3.3 is not satisfied. Suppose there are three funds with the rate of returns r_1, r_2 and r_3, respectively, on a given time period $[s, s+1]$. Then

$$\bar{r}_{PL}(s, s+1) = \sum_{i=1}^{3} \frac{1}{2} r_i \left(\frac{A_i(s)}{A(s)} + \frac{A_i(s+1)}{A(s+1)} \right),$$

where

$$A(s+1) = \sum_{i=1}^{3} A_i(s+1).$$

Let us group the first fund and the second one. In this way there are two funds: the first one has the return

$$r_1^g = \sum_{i=1}^{2} \frac{1}{2} r_i \left(\frac{A_i(s)}{\sum_{i=1}^{2} A_i(s)} + \frac{A_i(s+1)}{\sum_{i=1}^{2} A_i(s+1)} \right),$$

and the return of the second one equals $r_2^g = r_3$. By definition (3.1),

$$\bar{r}_{PL}^g = \frac{1}{2} r_i^g \left(\frac{\sum_{i=1}^{2} A_i(s)}{A(s)} + \frac{\sum_{i=1}^{2} A_i(s+1)}{A(s+1)} \right) + \frac{1}{2} r_2^g \left(\frac{A_3(s)}{A(s)} + \frac{A_3(s+1)}{A(s+1)} \right).$$

After an easy algebra we get

$$\begin{aligned} \bar{r}_{PL}^g = & \sum_{i=1}^{2} \frac{1}{4} r_i \left(A_i(s) + A_i(s+1) \right) \left(\frac{1}{A(s)} + \frac{1}{A(s+1)} \right) + \\ & + \frac{1}{2} r_3 \left(\frac{A_3(s)}{A(s)} + \frac{A_3(s+1)}{A(s+1)} \right). \end{aligned}$$

Clearly $\bar{r}_{PL}^g \neq \bar{r}_{PL}$. For example, if we assume that the number of units of each fund is constant and if $A_1(s) = 10^6$, $A_1(s+1) = 1.1 \cdot 10^6$, $A_2(s) = 3 \cdot 10^6$, $A_2(s+1) = 3.21 \cdot 10^6$, $A_3(s) = 4 \cdot 10^6$, $A_3(s+1) = 4.19 \cdot 10^6$, then

$$\bar{r}_{PL} = 6.26\%, \quad \bar{r}_{PL}^g = 7.48\%.$$

Changing $A_3(s+1) = 4.19 \cdot 10^6$ for $A_3(s+1) = 4.39 \cdot 10^6$, we get

$$\bar{r}_{PL} = 8.76\%, \quad \bar{r}^g_{PL} = 7.47\%.$$

Hence the average return $\bar{r}_{PL}$ may increase or decrease after grouping the funds. In our opinion, the fact that $\bar{r}_{PL}$ has not Property 3.3 makes it useless. Indeed, using it in Poland to report on pension funds investment efficiency leads to several practical problems described in [16]. A more detailed consideration on grouping funds is placed in Section 4.

Property 3.4. *If on a subset of a probability space the accounting units of all funds have the same values over $[s,t]$, i.e. $w_1(u) = w_2(u) = \ldots = w_n(u)$ for every $s \le u \le t$, then*

$$\bar{r}_A(s,t) = \frac{w_1(t) - w_1(s+)}{w_1(s+)}$$

holds on the same subset.

Property 3.5. *Suppose that $k_i(u) = \alpha_i \phi(u)$ for all $u \in [s,t]$, $i = 1, 2, \ldots, n$, where $\alpha_i \ge 0$, $\sum_{i=1}^n \alpha_i = 1$ and $\phi : [s,t] \to (0, \infty)$. Then*

$$\bar{r}_A(s,t) = \sum_{i=1}^n A_i^*(s) r_i(s,t),$$

where $r_i(s,t) = (w_i(t) - w_i(s+))/w_i(s+)$.

Observe that the above formula is equivalent to formula (12) of Gajek and Kałuszka (2001). If the number of units of every fund is constant over the time interval $[s,t]$, i.e. $\phi(u) = 1$ for all u, then

$$\bar{r}_A(s,t) = \frac{A(t) - A(s)}{A(s)}.$$

Property 3.6. *For every $s < t$*

$$\min_{1 \le i \le n, s \le u < t} r_i(u, u+1) \le \bar{r}_A(s,t) \le \max_{1 \le i \le n, s \le u < t} r_i(u, u+1).$$

The next property says that the influence of small funds on the average return of the group of funds is asymptotically negligible.

Property 3.7. *Given k and a fixed elementary event, if* $\max_{i\neq k} A_i(u) \leq \theta A_k(u)$, $u \in [s,t)$, *then*

$$\bar{r}_A(s,t) = \frac{w_k(t) - w_k(s)}{w_k(s)} + \delta(\theta),$$

where $\delta(\theta) \to 0$ *as* $\theta \to 0$.

Observe that the following definition of average rate of return does not possess Property 3.7:

$$\bar{r}_V(s,t) = [(1 + r_1(s,t)) \cdot \ldots \cdot (1 + r_n(s,t))]^{1/n} - 1,$$

where $r_i(s,t)$ is defined by (2.14). The average return $\bar{r}_V(s,t)$ is a counterpart of the well-known Value Line Composite Index (VLIC index) since

$$\bar{r}_V(s,t) = \left(\prod_{i=1}^{n} \frac{w_i(t)}{w_i(s)}\right)^{1/n} - 1.$$

The next property says that if we move some assets from a less effective fund to a more effective one, then the average rate of return increases.

Property 3.8. *Let $s < u < t$. Suppose that $w_i(s) = w_i(u) = w_i(t)$ for every $i = 3, 4, \ldots, n$ and suppose $r_1(s,u) < r_2(s,u)$. Moreover, suppose that some clients transfer their assets from the first fund to the second one at time u. Then the average return increases over the time interval $[s,t]$ if and only if $r_1(u,t) < r_2(u,t)$.*

4. Merger of funds

Suppose that there exists n pension funds at time $t = 0, 1, 2, \ldots, \tau$. At time τ the n-th fund and the $(n-1)$-th one form a new fund, say $(n-1)$-th. The assets of the new fund are equal to

$$k_{n-1}(\tau) w_{n-1}(\tau) + k_n(\tau) w_n(\tau).$$

At time τ, the number of units of the new fund will be denoted by $k_{n-1}(\tau+)$. The value of one unit of the new fund, $w_{n-1}(\tau+)$, is calculated according to

the formula

$$k_{n-1}(\tau+)w_{n-1}(\tau+) = k_{n-1}(\tau)w_{n-1}(\tau) + k_n(\tau)w_n(\tau).$$

Suppose that up to time $T > \tau$, the number of funds is constant. How to calculate the average return over the time period $[0, T]$?

Observe that the merger of the above two funds can be treated as an allocation of all assets of the n-th fund to the $(n-1)$-th one. After allocation, the units of the $(n-1)$-th fund split so that the new value of the unit of the $(n-1)$-th fund is equal to $w_{n-1}(\tau+)$ and the number of units is equal to $k_{n-1}(\tau+)$. By (2.13),

$$\begin{aligned}\bar{r}_A(0,T) = &\prod_{t=0}^{\tau-1}\left(\sum_{j=1}^{n} A_j^*(t)\frac{w_j(t+1)}{w_j(t)}\right)\times \\ &\times\left(\sum_{j=1}^{n-2} A_j^*(\tau)\frac{w_j(\tau+1)}{w_j(\tau)} + A_{n-1}^*(\tau)\frac{w_{n-1}(\tau+1)}{w_{n-1}(\tau+)}\right)\times \\ &\times\prod_{t=\tau+1}^{T-1}\left(\sum_{j=1}^{n-2} A_j^*(t)\frac{w_j(t+1)}{w_j(t)} + A_{n-1}^*(t)\frac{w_{n-1}(t+1)}{w_{n-1}(t)}\right) - 1,\end{aligned} \tag{4.1}$$

provided there is no split of units of others funds up to the time T. The rate of return $r'_{n-1}(0,\tau)$ of the new $(n-1)$-th fund at the moment τ equals

$$\begin{aligned}r'_{n-1}(0,\tau) = &\prod_{t=0}^{\tau-1}\left(\frac{A_{n-1}(t)}{A_{n-1,n}(t)}\frac{w_{n-1}(t+1)}{w_{n-1}(t)}+\right. \\ &\left.+\frac{A_n(t)}{A_{n-1,n}(t)}\frac{w_n(t+1)}{w_n(t)}\right) - 1,\end{aligned} \tag{4.2}$$

where by

$$A_{n,n-1}(t) = k_{n-1}(t)w_{n-1}(t) + k_n(t)w_n(t)$$

we denote the total assets at time t of funds numbered n and $n-1$.

Example 4.1. Suppose at time $t = 0$ there are five funds with the following number of units and values of units:

$$k_1(0) = 10^6, k_2(0) = 9\cdot 10^5, k_3(0) = 4\cdot 10^5, k_4(0) = 3\cdot 10^5, k_5(0) = 2\cdot 10^5,$$

and

$$w_1(0) = 10.5, w_2(0) = 9.4, w_3(0) = 4.3, w_4(0) = 5, w_5(0) = 8.5$$

(in PLN). At the moment $t = 1$,

$$k_1(1) = 10^6, k_2(1) = 9.2 \cdot 10^5, k_3(1) = 4.3 \cdot 10^5, k_4(1) = 3 \cdot 10^5, k_5(1) = 2.2 \cdot 10^5,$$

and

$$w_1(1) = 10.8, w_2(1) = 9.7, w_3(1) = 4.4, w_4(1) = 5.5, w_5(1) = 8.6.$$

At time $\tau = 1$ fund No. 4 merge with fund No. 5. The new fund has $k_4(1+) = 6 \cdot 10^5$ units so the value of one unit equals

$$w_4(1+) = \frac{k_4(1)w_4(1) + k_5(1)w_5(1)}{k_4(1+)} = 5.9.$$

Assume that at time $t = 2$:

$$k_1(2) = 1.2 \cdot 10^6, k_2(2) = 9.4 \cdot 10^5, k_3(2) = 4.3 \cdot 10^5, k_4(2) = 6.1 \cdot 10^5,$$

$$w_1(2) = 10.9, w_2(2) = 9.6, w_3(2) = 4.4, w_4(2) = 6.2.$$

Then the average return on the time period $[0, 2]$ equals

$$\bar{r}_A(0,2) = \left(\sum_{j=1}^{5} A_j^*(0) \frac{w_j(1)}{w_j(0)} \right) \left(\sum_{j=1}^{3} A_j^*(1) \frac{w_j(2)}{w_j(1)} + A_4^*(1) \frac{w_4(2)}{w_4(1+)} \right) - 1,$$

(see (4.1), so $\bar{r}_A(0,2) = 0.040384 = 4\%$.

By (4.2) the rate of return of the new fund at $\tau = 1$ equals 5.312%. Observe that the arithmetic mean of the rate of return of fund No. 4 and fund No. 5 at time $\tau = 1$ is equal to $(10\% + 1.117\%)/2 = 5.58\%$ and is greater than the rate of return of the new fund at the same time.

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